Elemente de GA
10.04.2020

Pufene Alex 0.5p
Cojanu Adina 1p+0.5p+0.5p
Mariuca Andrei 0.5p

Se vede?

Vectori. Probleme tratate vectorial

Def1 Fie A şi B două puncte în plan.
Perechea ordonată (A, B) se numeşte segment orientat sau vector legat în punctul A şi se notează cu $\overrightarrow{AB}$.

originea lui $\overrightarrow{AB}$ — extremitatea segmentului orientat $\overrightarrow{AB}$

Def2 Prin modulul lui $\overrightarrow{AB}$ se înţelege lungimea segmentului neorientat $[AB]$ (şi se notează $|\overrightarrow{AB}|$)

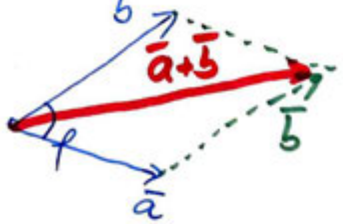
Def3 Prin vectorul liber $\overline{AB}$ se înţelege un vector $\bar{v}$ cu prop. $|\bar{v}| = |\overrightarrow{AB}|$, de aceeaşi direcţie şi acelaşi sens cu $\overrightarrow{AB}$.

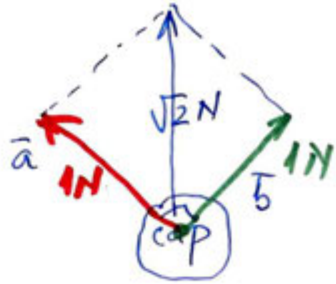
Def4 Pentru 2 vectori liberi $\bar{a}$ şi $\bar{b}$ se def.
adunarea $\bar{a}+\bar{b}$ după regula paralelogramului.

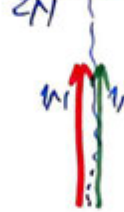
$$\bar{a}\cdot\bar{b} = |\bar{a}|\cdot|\bar{b}|\cdot\cos\varphi$$

↓

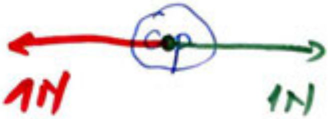
produsul scalar al vectorilor $\bar{a}$ şi $\bar{b}$







Răileanu Bogdan 0.5
Untarche Vlad 0.5
Cocor Cristina 0.5
Cantuniar Mădălina 0.5
Constantinescu 1p

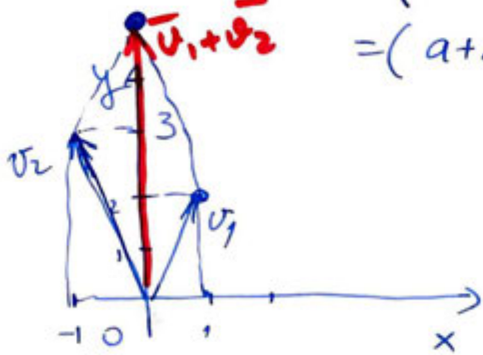


$\bar{v}_1 = (1,2)$
$v_2 = (-1,3)$

$\rightsquigarrow$ suma $\underset{(a,b)}{\bar{v}_1} + \underset{(c,d)}{\bar{v}_2} = (1,2)+(-1,3) =$???

$= (1+(-1),\ 2+3) = (0,5)$

$= (a+c,\ b+d)$



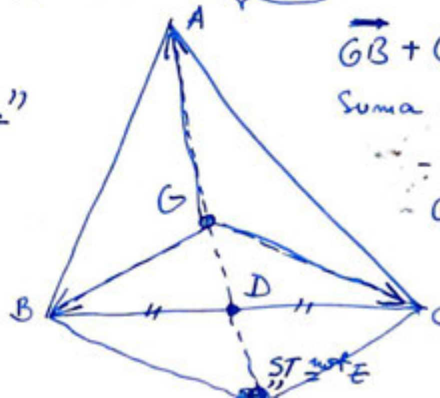
Puflene Alex 1p
Marinescu Andreea 1p + 0.5p + 0.5p

Pb1 G = centrul de greutate al $\Delta ABC \Leftrightarrow$

$\Leftrightarrow \overline{GA} + \overline{GB} + \overline{GC} = \bar{0}$

"$\Rightarrow$"

Temă "$\Leftarrow$"



$\overrightarrow{GB} + \overline{GC} = \overline{GE}$ (GBEC par)

Suma devine: $\overline{GA} + \overline{GE} = \bar{0}$

- aceleași lungimi?
- G = centru de greutate $\Rightarrow$

$\Rightarrow \left.\begin{array}{l} GD = \frac{1}{3} AD \\ DE = GD \end{array}\right\} \Rightarrow$

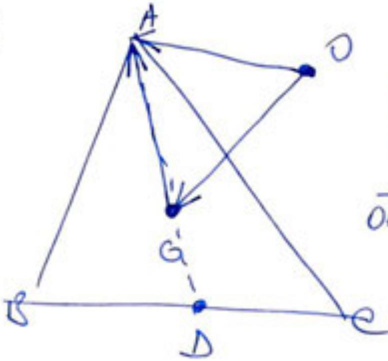
$\Rightarrow GE = \frac{2}{3} AD = AG.$ □

O un punct în plan

Pb2 G = centru de greutate, $O \in \Pi$ (planului)

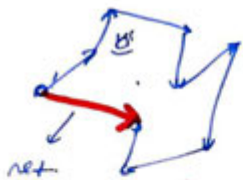
$$\overline{OA} + \overline{OB} + \overline{OC} = 3\overline{OG}$$

Soluție:



$$\left.\begin{aligned} \overline{OA} &= \overline{OG} + \overline{GA} \\ \overline{OB} &= \overline{OG} + \overline{GB} \\ \overline{OC} &= \overline{OG} + \overline{GC} \end{aligned}\right| \overset{(+)}{\Rightarrow}$$

$$\overline{OA} + \overline{OB} + \overline{OC} = 3\overline{OG} + \underbrace{\overline{GA} + \overline{GB} + \overline{GC}}_{\overset{Pb1}{=} 0} . \blacksquare$$

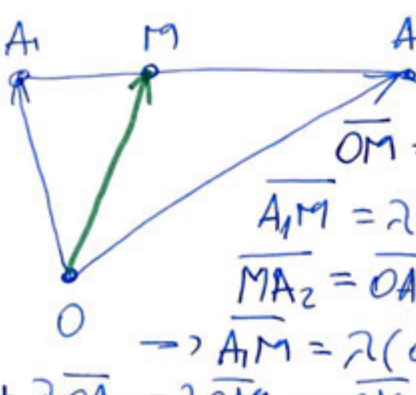


Pb3 Fie A_1A_2 și M a.î. $\dfrac{A_1M}{MA_2} = \lambda$, $O \in \Pi$.

S.s.a.c. $\boxed{\overline{OM} = \dfrac{\overline{OA_1} + \lambda \overline{OA_2}}{1+\lambda}}$

Sol:

Cojanu Adina arp



$$\overline{OM} = \overline{OA_1} + \overline{A_1M}$$

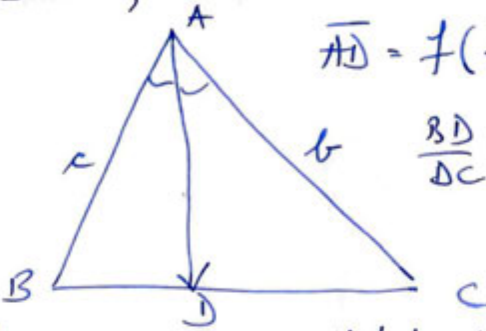
$$\left.\begin{aligned} \overline{A_1M} &= \lambda \overline{MA_2} \\ \overline{MA_2} &= \overline{OA_2} - \overline{OM} \end{aligned}\right| \Rightarrow$$

$$\Rightarrow \overline{A_1M} = \lambda(\overline{OA_2} - \overline{OM})$$

$$\overline{OM} = \overline{OA_1} + \lambda \overline{OA_2} - \lambda \overline{OM} \Rightarrow \overline{OM}(1+\lambda) = \overline{OA_1} + \lambda \overline{OA_2} . \square$$

Tema 2 (EGA) ΔABC, AD bis

(2p)



$\overline{AD} = f(\overline{AB}, \overline{AC})$

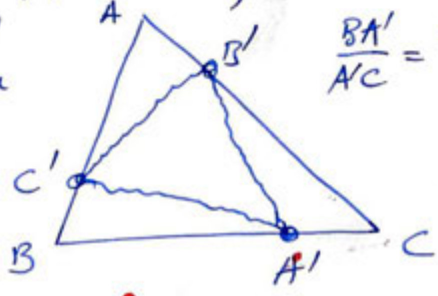
$\frac{BD}{DC} = \frac{c}{b}$

(2p)

Tema 3 (EGA). Fie ΔABC, $\Delta A'B'C'$ astfel:

$\frac{BA'}{A'C} = \frac{CB'}{B'A} = \frac{AC'}{C'B} = k$

ΔABC și $\Delta A'B'C'$ au același centru de greutate.



Ivan Andrei 6p

cityinnohub@univ-ovidius.ro

MOF BAC - MI

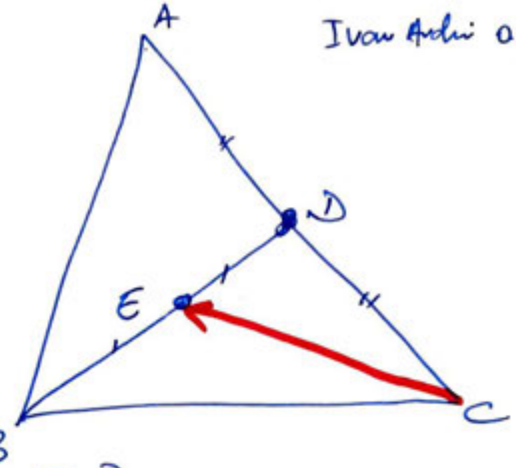
Conducmcu Mădălina 0.5 p,
Marinescu Holm 0.5 p

(I 5.) ΔABC, D = mij AC, E = mij BD

$$\overline{CE} = \frac{1}{4}\overline{CA} - \frac{1}{2}\overline{BC}$$

Sol

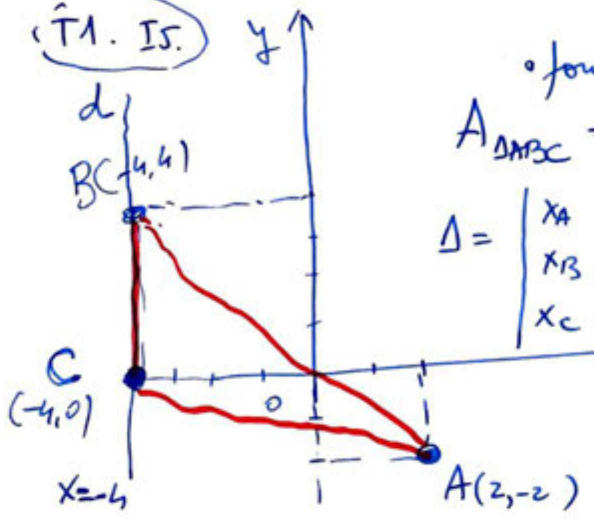
Ivan Andi 0.5p + 0.5p



$$\left.\begin{array}{l}\overline{CE} = \dfrac{\overline{CB}+\overline{CD}}{2} \\ \overline{CD} = \dfrac{\overline{CA}}{2}\end{array}\right\} \Rightarrow \overline{CE} = \frac{\overline{CB}}{2} + \frac{\overline{CA}}{4} =$$

$$= \frac{1}{4}\overline{CA} - \frac{1}{2}\overline{BC} \;.\;\square$$

(T1. I5.)



• formulă

$$A_{\Delta ABC} = \frac{1}{2}|\Delta|$$

$$\Delta = \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} = \begin{vmatrix} 2 & -2 & 1 \\ -4 & 4 & 1 \\ -4 & 0 & 1 \end{vmatrix} =$$

$$= (-4)\begin{vmatrix} -2 & 1 \\ 4 & 1 \end{vmatrix} + 1\cdot\begin{vmatrix} 2 & -2 \\ -4 & 4 \end{vmatrix} =$$

$$= (-4)\cdot(-6) + 0 = 24$$

$$A_{\Delta ABC} = \frac{1}{2}\cdot 24 = 12$$

$$x + \quad 4 = 0$$

$$d(M_0, d) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$$

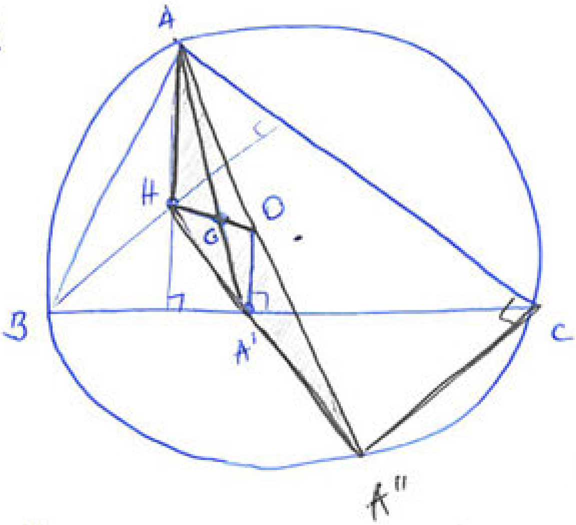
$$(+3, 2-2)$$

$$d(A, BC) = \frac{|1 \cdot 2 + 0 \cdot (-2) + 4|}{\sqrt{1^2+0^2}} = 6$$

$$A_{\Delta ABC} = \frac{BC \cdot d(A, BC)}{2} = \frac{4 \cdot 6}{2} = 12$$

(8) Să se arate că într-un ΔABC, O, G, H coliniare. (dreapta lui Euler)

Soluție: Sol 1



A'' = diam opus lui A

$BHCA''$ par $\Rightarrow A' = $ mij HA''; $OA' \parallel AH$ $\Rightarrow$

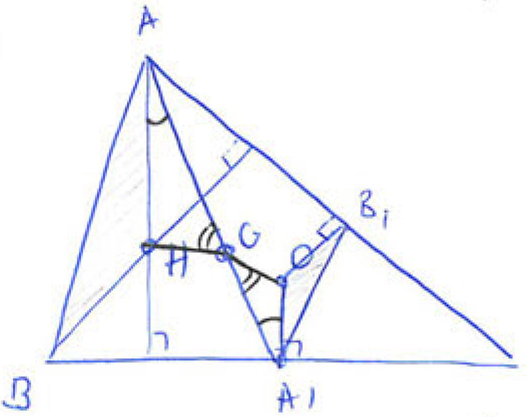
$\Rightarrow OA' = \frac{1}{2} AH$

Fie $\{G\} = HO \cap AA'$

$\Delta AHG \sim \Delta A'OG \Rightarrow$

$\Rightarrow GA' = \frac{1}{2} GA \Rightarrow G$ = centrul de greutate. □

Sol 2



$\frac{OA_1}{AH} = \frac{1}{2}$, $\frac{A_1G}{GA} = \frac{1}{2}$ $\Rightarrow \frac{OA_1}{AH} = \frac{A_1G}{GA}$, $\hat{A_1} = \hat{A}$ $\Rightarrow$

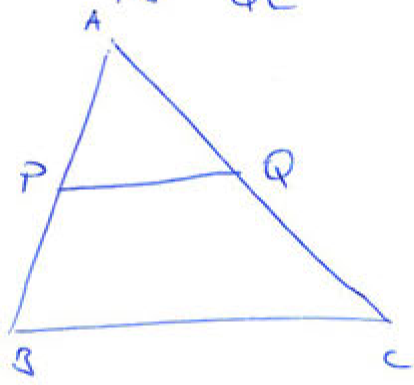
$\Rightarrow \Delta AHG \sim \Delta A_1GO \Rightarrow$

$\widehat{HGA} = \widehat{A_1GO} \Rightarrow H, G, O$ coliniare.

Sol 3

pb 2: $3\overline{OG} = \overline{OA} + \overline{OB} + \overline{OC}$; pb 7: $\overline{OH} = \overline{OA} + \overline{OB} + \overline{OC}$ $\Rightarrow \overline{OH} = 3\overline{OG} \Rightarrow O, G, H$ coliniare.

(9) Teorema Thales. Într-un ΔABC $PQ \parallel BC \Leftrightarrow \frac{AP}{PB} = \frac{AQ}{QC}$



Soluție: "$\Longrightarrow$" $PQ \parallel BC \Rightarrow \exists \lambda$ ai $\overline{PQ} = \lambda \overline{BC}$; $\overline{BC} = \overline{BA} + \overline{AC}$ $\Rightarrow$

$\Rightarrow \overline{PQ} = \lambda(\overline{BA} + \overline{AC}) = \lambda \overline{BA} + \lambda \overline{AC}$ (1)

Fie $r_1 = \frac{PA}{PB}$, $r_2 = \frac{QA}{QC}$. Tb. $r_1 = r_2$

$\overline{PA} = r_1 \overline{PB}$ și $\overline{QA} = r_2 \overline{QC}$

$\overline{PB} = \overline{AB} - \overline{AP}$ $\Rightarrow \overline{PA} = r_1(\overline{AB} - \overline{AP}) = r_1 \overline{AB} + r_1 \overline{PA} \Rightarrow$

$\Rightarrow (1 - r_1)\overline{PA} = r_1 \overline{AB} \Rightarrow \boxed{\overline{PA} = \frac{r_1}{1 - r_1} \overline{AB}}$

Analog $\boxed{\overline{AQ} = \frac{r_2}{r_2 - 1} \overline{AC}}$

Dar, $\overline{PQ} = \overline{PA} + \overline{AQ} = \frac{r_1}{1 - r_1} \overline{AB} + \frac{r_2}{r_2 - 1} \overline{AC} = \frac{r_1}{r_1 - 1} \overline{BA} + \frac{r_2}{r_2 - 1} \overline{AC}$ (2)

Din ① + ② $\Rightarrow \frac{r_1}{r_1-1} = \frac{r_2}{r_2-1} \Rightarrow r_1r_2 - r_2 = r_1r_2 - r_1 \Rightarrow r_1 = r_2$. □

"$\Leftarrow$" Pp $\frac{AP}{PB} = \frac{AQ}{QC} = t$

Dar $\overline{PQ} = \overline{PA} + \overline{AQ} = \frac{t}{t-1}\overline{BA} + \frac{t}{t-1}\overline{AC} = \frac{t}{t-1}(\overline{BA}+\overline{AC}) = \frac{t}{t-1}\overline{B}$

$\Rightarrow \overline{PQ} = \frac{t}{t-1}\overline{BC} \Rightarrow \overline{PQ} \parallel \overline{BC}$. □

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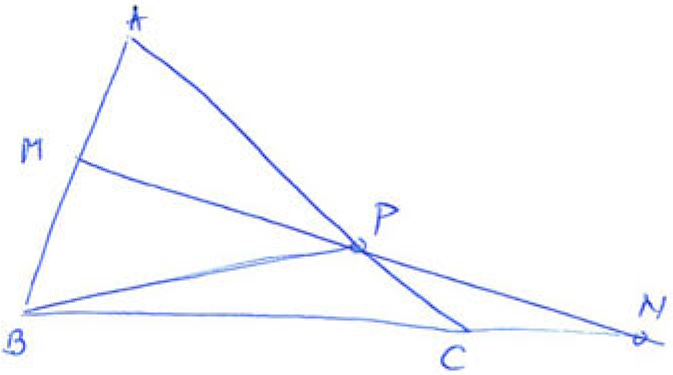
(10) <u>Teorema Menelau</u> M, N, P coliniare $\Leftrightarrow \frac{\overline{MA}}{\overline{MB}} \cdot \frac{\overline{NB}}{\overline{NC}} \cdot \frac{\overline{PC}}{\overline{PA}} = 1$

$-p \quad q \quad -r$

Tb $\boxed{pqr = 1}$



<u>Soluție</u> M, N, P coliniare $\Leftrightarrow \overline{MN} = \alpha\,\overline{MP}$ $\quad \frac{-1}{1+r}\ \frac{1}{1+r} =$

$\overline{MN} = \overline{BN} - \overline{BM}$

$\overline{MP} = \overline{BP} - \overline{BM}$

$\frac{\overline{MA}}{\overline{MB}} = -p \Rightarrow \frac{\overline{MA}}{\overline{BM}} = p \Rightarrow \frac{\overline{BM}+\overline{MA}}{\overline{BM}} = \frac{1+p}{1} \Rightarrow \frac{\overline{BA}}{\overline{BM}} = \frac{1+p}{1} \Rightarrow$

$\Rightarrow \boxed{\overline{BM} = \frac{1}{1+p}\cdot\overline{BA}}$

$\frac{\overline{NB}}{\overline{NC}} = q \Rightarrow \frac{\overline{NB}}{\overline{NB}-\overline{CB}} = q \Rightarrow \overline{NB} = q\,\overline{NB} - q\,\overline{CB} \Rightarrow$

$\Rightarrow \overline{NB}\,(1-q) = -q\,\overline{CB} \Rightarrow \boxed{\overline{BN} = \frac{q}{1-q}\,\overline{CB}}$

$\overline{BP} = \overline{BA} + \overline{AP}$

$\overline{AP}$? $\frac{\overline{PC}}{\overline{PA}} = -r \Rightarrow \frac{\overline{PC}}{\overline{AP}} = r \Rightarrow \frac{\overbrace{\overline{AP}+\overline{PC}}^{\overline{AC}}}{\overline{AP}} = \frac{1+r}{1} \Rightarrow \boxed{\overline{AP} = \frac{1}{1+r}\,\overline{AC}}$

$\overline{BP} = \overline{BA} + \overline{AP} = \overline{BA} + \frac{1}{1+r}\,\underbrace{\overline{AC}}_{\overline{BC}-\overline{BA}} = \overline{BA} + \frac{1}{1+r}(\overline{BC} - \overline{BA}) =$

$= \overline{BA} + \frac{1}{1+r}\overline{BC} - \frac{1}{1+r}\overline{BA} = \boxed{\frac{1}{1+r}\,\overline{BC} + \frac{r}{1+r}\,\overline{BA} = \overline{BP}}$

$$\overline{MN} = \overline{BN} - \overline{BM} = \frac{q}{1-q}\,\overline{CB} - \frac{1}{1+p}\,\overline{BA}$$

$$\overline{MP} = \overline{BP} - \overline{BM} = \frac{1}{1+r}\,\overline{BC} + \frac{r}{1+r}\,\overline{BA} - \frac{1}{1+p}\,\overline{BA} = \frac{1}{1+r}\,\overline{BC} + \left(\frac{r}{1+r} - \frac{1}{1+p}\right)\overline{BA}$$

M, N, P col $\iff$ raportul coef ~~...~~ este acelaşi:

$$\frac{\frac{q}{q-1}}{\frac{1}{1+r}} = \frac{-\frac{1}{1+p}}{\frac{r}{1+r}-\frac{1}{1+p}} \iff \frac{q}{1-q}\left(\frac{r}{1+r} - \frac{1}{1+p}\right) = \frac{1}{1+r}\,\frac{1}{1+p}$$

$$\frac{q}{1-q}\,\frac{r+pr-1-r}{(1+p)(1+r)} = \frac{1}{(1+r)(1+p)} \iff \frac{q}{1-q}(pr-1) = 1$$

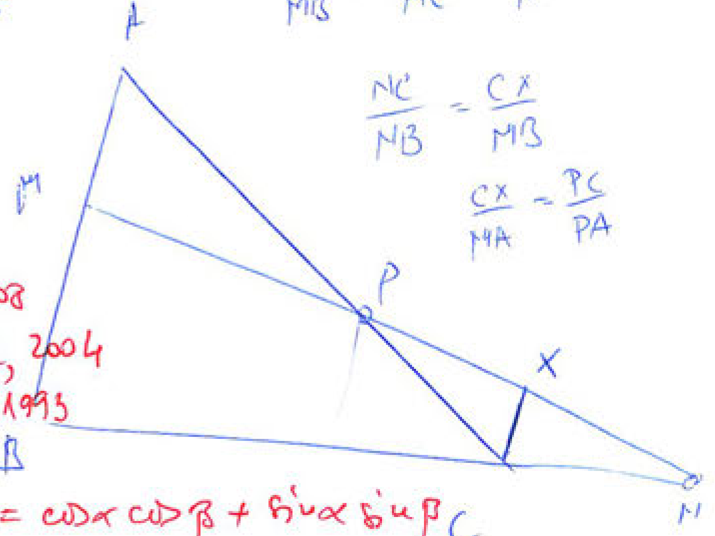
$$pqr - q = 1 - q \iff \boxed{pqr = 1}$$

~~din $pqr = 1 \Rightarrow r = \frac{1}{pq}$~~

$\frac{MA}{MB}\cdot\frac{NB}{NC}\cdot\frac{PC}{PA} = 1$

$\frac{NC}{NB} = \frac{CX}{MB}$

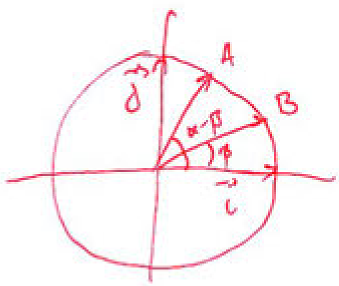
$\frac{CX}{MA} = \frac{PC}{PA}$



Bib:

1. M. Ciucă, Seminar Geometrie II, site, 2008
2. Gh. Atanasiu, Alg lin ....., ed All, 2004
3. N. Donciu, Geom. an..., EDP, 1993

Să se stab. vect formula $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$



$\overline{OA}\cdot\overline{OB} = \cos(\alpha-\beta)$

$\overline{OA} = \cos\alpha\cdot\overline{i} + \sin\alpha\cdot\overline{j}$

$\overline{OB} = \cos\beta\cdot\overline{i} + \sin\beta\cdot\overline{j}$