Lectii pregatire Bac 2021
Matrice, determinanti si sisteme de ecuatii liniare
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10 aprilie 2021

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I. Sisteme compatibil determinate

Exercitiul nr. 1: Sa se rezolve sistemul de ecuatii liniare:

$$\begin{cases} x+y+2z=-1 \\ 2x-y+4z=-4 \\ 4x+y+4z=-2 \end{cases}$$

Soluție:

$$A=\begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 4 \\ 4 & 1 & 4 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$$

$\text{rang}\, A \leq 3$; $\Delta = \det A$;

$$\det A = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 4 \\ 4 & 1 & 4 \end{vmatrix} = -4+4+16-(-8+4+8) = 12 \neq 0 \Rightarrow \text{rg} A = 3 \Rightarrow$$

$\boxed{\Delta = 12}$

$\Rightarrow$ sistem comp. det (Cramer) (sol unică)

$$\Delta_x = \begin{vmatrix} -1 & 1 & 2 \\ -4 & -1 & 4 \\ -2 & 1 & 4 \end{vmatrix} = 4-8-8-(4-4-16) = -12+16 = 4 \Rightarrow x = \frac{\Delta x}{\Delta} \Rightarrow$$

$\Rightarrow x = \frac{4}{12} \Rightarrow \boxed{x = \frac{1}{3}}$

$$\Delta_y = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -4 & 4 \\ 4 & -2 & 4 \end{vmatrix} = -16-8-16-(-32-8-8)$$

$= -16-8-16+32+8+8$

$= -24+32 = 8$

$y = \frac{\Delta y}{\Delta} \Rightarrow y = \frac{8}{12} \Rightarrow \boxed{y = \frac{2}{3}}$

$$\Delta_z = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & -4 \\ 4 & 1 & -2 \end{vmatrix} = 2-2-16-(4-4-4) = -12$$

$z = \frac{\Delta z}{\Delta} = \frac{-12}{12} \Rightarrow \boxed{z=-1}$

$S = \left\{ \left(\frac{1}{3}, \frac{2}{3}, -1 \right) \right\}$

II. Rangul unei matrice

Exercitiul nr. 2: Calculati rangul matricelor de mai jos:

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 3 & -1 \\ 2 & 4 & 0 \end{pmatrix} ; \quad \bar{A} = \begin{pmatrix} 2 & 1 & -1 & 1 \\ 3 & 3 & -1 & 2 \\ 2 & 4 & 0 & 3 \end{pmatrix}$$

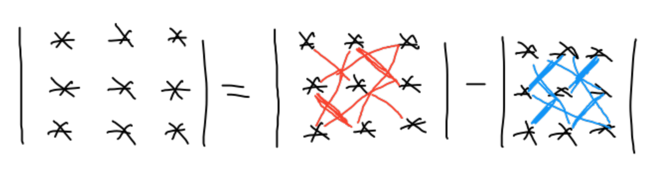
$\text{rang}(A) = \text{rg}(A) \leq 3$

$\Delta_p = \begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix} = 6-3 = 3 \neq 0 \Rightarrow$

$\Rightarrow \text{rg} A \geq 2$

$$\det A = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 3 & -1 \\ 2 & 4 & 0 \end{vmatrix} = 0-12-2-(-6-8+0) = -14+14 = 0$$

$\Rightarrow \boxed{\text{rg} A = 2}$ (1)



Regula triunghiului (pentru calculul determinantilor de ordin

$$\bar{A} = \begin{pmatrix} 2 & 1 & -1 & 1 \\ 3 & 3 & -1 & 2 \\ 2 & 4 & 0 & 3 \end{pmatrix} \Rightarrow \text{rg} \bar{A} \leq 3$$

$\Delta_p = \begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix} = 3 \neq 0 \Rightarrow \text{rg} \bar{A} \geq 2$

$$\Delta_1 = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 3 & -1 \\ 2 & 4 & 0 \end{vmatrix} = \det A = 0$$

$$\Delta_{car} = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 3 & 2 \\ 2 & 4 & 3 \end{vmatrix} = 18+12+4-(6+16+9) = 34-31 = 3 \neq 0 \Rightarrow$$

$\Rightarrow \boxed{\text{rg} \bar{A} = 3}$ (2)

III. Sisteme incompatibile

Exercitiul nr. 3: Sa se arate ca sistemul de mai jos nu are solutii:

$$\begin{cases} 2x+y-z=1 \\ 3x+3y-z=2 \\ 2x+4y=3 \end{cases}$$

Solutie: $A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 3 & -1 \\ 2 & 4 & 0 \end{pmatrix}$; $\bar{A} = \begin{pmatrix} 2 & 1 & -1 & 1 \\ 3 & 3 & -1 & 2 \\ 2 & 4 & 0 & 3 \end{pmatrix}$

Teorema (Kronecker - Capelli): Un sistem liniar este compatibil daca si numai daca rangul matricei sistemului coincide cu rangul matricei extinse.

Sistem compatibil (are sol) $\Leftrightarrow \text{rg} A = \text{rg} \bar{A}$

– din (1) și (2) $\Rightarrow \underbrace{\text{rg} A}_{2} \neq \underbrace{\text{rg} \bar{A}}_{3}$ $\Rightarrow$

$\Rightarrow$ sistemul este incompatibil (nu are soluții)

IV. Sisteme compatibile simplu nedeterminate

Exercitiul nr. 4: Sa se rezolve sistemul:

$$\begin{cases} 2x-y+3z=4 \\ 3x+4y-z=-5 \\ x+5y-4z=-9 \end{cases}$$

Solutie:

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 3 & 4 & -1 \\ 1 & 5 & -4 \end{pmatrix} ; \quad \bar{A} = \begin{pmatrix} 2 & -1 & 3 & 4 \\ 3 & 4 & -1 & -5 \\ 1 & 5 & -4 & -9 \end{pmatrix}$$

• $\text{rg} A \leq 3$

$\Delta_p = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 8+3 = 11 \neq 0 \Rightarrow \text{rg} A \geq 2$

$$\det A = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 4 & -1 \\ 1 & 5 & -4 \end{vmatrix} = -32+45+1-(12-10+12) = 14-14 = 0 \Rightarrow \boxed{\text{rg} A = 2} \ (3)$$

• $\text{rg} \bar{A} \geq 2$; $\text{rg} \bar{A} \leq 3$

$\Delta_1 = \det A = 0$

$$\Delta_{car} = \begin{vmatrix} 2 & -1 & 4 \\ 3 & 4 & -5 \\ 1 & 5 & -9 \end{vmatrix} = -72+60+5-(16-50+27) = -7-(-7) = 0$$

$\Rightarrow \boxed{\text{rg} \bar{A} = 2}$ (4)

– din (3), (4) $\Rightarrow \text{rg} A = \text{rg} \bar{A} = 2 \Rightarrow$

$\Rightarrow$ sistem compatibil nedeterminat simplu

- ecuatii principale: ecuatia 1, ecuatia 2;
- ecuatii secundare: ecuatia 3;
- necunoscute principale: x, y;
- necunoscute secundare: z ; $\boxed{z = \alpha}$, $\alpha \in \mathbb{R}$

$$\begin{cases} 2x-y = 4-3\alpha & |\cdot(4) \\ 3x+4y = -5+\alpha & \end{cases} \Leftrightarrow \begin{cases} 8x-4y = 16-12\alpha \\ 3x+4y = -5+\alpha \end{cases}$$

$11x = 11-11\alpha$

$\Rightarrow \boxed{x = 1-\alpha} \Rightarrow \boxed{y = \alpha-2}$

$$\boxed{S = \{ (1-\alpha, \alpha-2, \alpha) \mid \alpha \in \mathbb{R} \}}$$

V. Discutia sistemelor de ecuatii liniare dupa parametru

Subiect simulare nationala Bac 2019

Exercitiul II. 1 Se consideră matricea

$$A(a) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & a \\ 2 & a & 4 \end{pmatrix}$$ și sistemul de ecuații liniare

$$\begin{cases} x+y+z=1 \\ x+2y+az=2 \\ 2x+ay+4z=3 \end{cases}, \quad a \in \mathbb{R}.$$

a) Arătați că $\det(A(a)) = a(3-a)$, $\forall a \in \mathbb{R}$

Soluție:

$$\det(A(a)) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & a \\ 2 & a & 4 \end{vmatrix} \overset{C_2-C_1}{\underset{C_3-C_1}{=}} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & a-1 \\ 2 & a-2 & 2 \end{vmatrix} =$$

$= 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 1 & a-1 \\ a-2 & 2 \end{vmatrix} = 2-(a-1)(a-2) =$

$= 2-(a^2-3a+2) = -a^2+3a = a(3-a)$. ■

b) Pentru a=0, demonstrati ca sistemul de ecuatii este incompatibil.

Solutie:

• $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 4 \end{pmatrix} \Rightarrow \det A = 0$

• $\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 \\ 2 & 0 & 4 & 3 \end{pmatrix}$; $\Delta_p = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0 \Rightarrow$

$\Rightarrow \text{rg} \bar{A} \geq 2$

$$\Delta_{car} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 0 & 3 \end{vmatrix} = 6+4-4-3 = 3 \neq 0$$

$\Rightarrow \text{rg} \bar{A} = 3 \neq \text{rg} A (=2) \Rightarrow$ sist. incompatib. ■