Ex: $f:(-1,+\infty)\to\mathbb{R}$, $f(x)=\dfrac{2x+3}{x+2}$

a) Ssec orice primitivă F a lui f pe $(-1,+\infty)$ este strict crescătoare pe $(-1,+\infty)$.

b) $\displaystyle\int_0^1 \frac{f(x)}{x+1} = ?$

c) $\displaystyle\lim_{x\to+\infty} \frac{\int_x^{2x} f(t)\,dt}{x} = ?$

Sol:

b) $$\int_0^1 \frac{2x+3}{(x+1)(x+2)}\,dx = \int_0^1 \left(\frac{1}{1+x}+\frac{1}{x+2}\right)dx$$
$$= \ln(1+x)\Big|_0^1 + \ln(2+x)\Big|_0^1 =$$
$$= \ln 2 + \ln 3 - \ln 2 = \ln 3.$$

a) $F'(x)=f(x)$, $\forall x>-1$.

pt $x>-1$: $\begin{array}{l} 2x+3 > -2+3 = 1 > 0 \\ x+2 > -1+2 = 1 > 0 \end{array} \Bigg| \Rightarrow$

$f(x)>0$ pe $(-1,+\infty)$ $\Rightarrow$ $F'(x)>0$ pe $(-1,+\infty)$

$\Rightarrow$ F strict crescătoare pe $(-1,+\infty)$.

c) $$\int_x^{2x} \frac{2t+3}{t+2}\,dt = \int_x^{2x}\left(\frac{2t+4}{t+2}-\frac{1}{t+2}\right)dt$$
$$= \int_x^{2x}\left(2-\frac{1}{t+2}\right)dt$$
$$= 2t\Big|_x^{2x} - \ln(2+t)\Big|_x^{2x}$$
$$= 2x - \big(\ln(2+2x)-\ln(2+x)\big)$$
$$= 2x - \ln\frac{2+2x}{2+x}.$$

$$\lim_{x\to+\infty}\frac{\int_x^{2x} f(t)\,dt}{x} = \lim_{x\to+\infty}\left(2-\frac{1}{x}\ln\frac{2+2x}{2+x}\right) = 2-0 = 2$$

$$\frac{1}{x}\ln\frac{2+2x}{2+x} = \frac{1}{x}\ln\frac{\cancel{x}\left(\frac{2}{x}+2\right)}{\cancel{x}\left(\frac{2}{x}+1\right)} = \frac{1}{x}\ln\frac{\frac{2}{x}+2}{\frac{2}{x}+1}$$
$$\to 0\cdot\ln\frac{2}{1} = 0 \text{ pt } x\to+\infty.$$

Ex: $f:(-2,2)\to\mathbb{R}$, $f(x)=x+\dfrac{x^2}{x^2-4}$.

a) $\displaystyle\int_1^{3/2}\left(f(x)-\frac{x^2}{x^2-4}\right)dx=\frac{5}{8}$.

b) $\displaystyle\int_{-1}^{1}\left(f(x)+f(-x)\right)dx=4(1-\ln 3)$

c) Determinați $a\in(0,\sqrt3)$ astfel încât $\displaystyle\int_a^{\sqrt3}\sqrt{x-f(x)}\,dx=\sqrt3-1$.

Sol:

a) $\displaystyle\int_1^{3/2}x\,dx=\frac{x^2}{2}\Big|_1^{3/2}=\frac{9}{8}-\frac12=\frac{9-4}{8}=\frac58$.

b) $\displaystyle\int_{-1}^{1}\left(\cancel{x}+\frac{x^2}{x^2-4}+\cancel{(-x)}+\frac{(-x)^2}{(-x)^2-4}\right)dx=2\int_{-1}^{1}\frac{x^2}{x^2-4}dx$

$\displaystyle=2\left[\int_{-1}^{1}\frac{\cancel{x^2-4}}{\cancel{x^2-4}}dx+4\int_{-1}^{1}\frac{dx}{x^2-4}\right]$

$\displaystyle=2\left[x\Big|_{-1}^{1}+4\int_{-1}^{1}\frac{dx}{(x-2)(x+2)}\right]$

$\displaystyle=2\left(2+\int_{-1}^{1}\left(\frac{1}{x-2}-\frac{1}{x+2}\right)dx\right)$

$\displaystyle=2\left(2+\ln|x-2|\Big|_{-1}^{1}-\ln|x+2|\Big|_{-1}^{1}\right)$

$=2\left(2+(\ln1-\ln3)-(\ln3-\ln1)\right)$

$=2(2-2\ln3)=4(1-\ln3)$

c) $\displaystyle\int\sqrt{x-f(x)}\,dx=\int\sqrt{\frac{x^2}{-x^2+4}}\,dx=\int\frac{\sqrt{x^2}}{\sqrt{4-x^2}}dx$

$\displaystyle=\int\frac{|x|}{\sqrt{4-x^2}}dx=\int\frac{x}{\sqrt{4-x^2}}dx$ pe $(0,2)$

$\displaystyle=-\frac12\int\frac{(4-x^2)'}{\sqrt{4-x^2}}dx=-\sqrt{4-x^2}$

Pt $a\in(0,\sqrt3)$, $\displaystyle\int_a^{\sqrt3}\sqrt{x-f(x)}\,dx=-\sqrt{4-x^2}\Big|_a^{\sqrt3}$

$=\sqrt{4-a^2}-\sqrt{4-3}$

$=\sqrt{4-a^2}-1$

$\sqrt{4-a^2}-1=\sqrt3-1\Rightarrow\sqrt{4-a^2}=\sqrt3\Rightarrow4-a^2=3$

$\Rightarrow a^2=4-3=1$

$\Rightarrow a\in\{\pm1\}$. Dar $a>0$ $\Rightarrow a=1\in(0,\sqrt3)$.

Ex: $f:(-1,+\infty)\to\mathbb{R}$, $f(x)=\frac{3x}{\sqrt{x+1}}$

a) $\int_1^2 x\sqrt{x+1}\, f(x)\,dx = 7$

b) $\int_0^1 f^2(x)\,dx = ?$

c) Știm $F:(-1,\infty)\to\mathbb{R}$, $F(x)=2(x+1)\sqrt{x+1}-6\sqrt{x+1}+4$ este primitivă pt f. De arătat că

$$\int_0^3 f(x)F(x)\,dx = 32.$$

Sol:

a) $\int_1^2 x\sqrt{\cancel{x+1}}\cdot \frac{3x}{\cancel{\sqrt{x+1}}}\,dx = \int_1^2 3x^2\,dx = x^3\Big|_1^2 = 8-1=7.$

b) $\int_0^1 \frac{9x^2}{x+1}\,dx = 9\left[\int_0^1 \frac{x^2-1}{x+1}\,dx + \int_0^1 \frac{dx}{x+1}\right]$

$= 9\left\{\int_0^1 \frac{(x-1)\cancel{(x+1)}}{\cancel{x+1}}\,dx + \ln(1+x)\Big|_0^1\right\}$

$= 9\left\{\frac{x^2}{2}\Big|_0^1 - x\Big|_0^1 + \ln 2\right\}$

$= 9\left(\ln 2 + \frac{1}{2} - 1\right) = 9\left(\ln 2 - \frac{1}{2}\right)$

c) $\int_0^3 f(x)F(x)\,dx = \int_0^3 F'(x)F(x)\,dx$

$= \frac{1}{2}\int_0^3 \left(F^2(x)\right)'\,dx$

$= \frac{1}{2}F^2(x)\Big|_0^3$

$= \frac{1}{2}\left(F(3)^2 - F(0)^2\right)$

$= \frac{1}{2}\left((2\cdot4\cdot2 - 6\cdot2+4)^2 - 0\right)$

$= \frac{1}{2}(8^2-0)$

$= 32.$

Ex: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 4x^3+1$.

a) $\int_0^1 f(x)\,dx = 2$

b) $\int_0^1 x^2 f(x)^3\,dx = ?$

c) $\lim\limits_{x\to\infty} \int_1^x \ln f(t)\,dt = +\infty$.

Sol:

a) $\int_0^1 (4x^3+1)\,dx = x^4\Big|_0^1 + x\Big|_0^1 = 1+1=2.$

b) $\int_0^1 x^2(4x^3+1)^3\,dx = \frac{1}{3}\int_0^1 (4t+1)^3\,dt$

$x^3 = t \Rightarrow 3x^2\,dx = dt$

$= \frac{1}{3\cdot 4}\int_1^5 s^3\,ds$

$4t+1 = s \Rightarrow 4\,dt = ds$

$= \frac{1}{12}\cdot \frac{s^4}{4}\Big|_1^5$

$= \frac{5^4-1}{48} = \frac{624}{48} = 13$

c) Pt $t \geq 1$ avem $f(t) = 4t^3+1 \geq 4+1 = 5$.

$\forall x \geq 1$, $\int_1^x \ln f(t)\,dt \geq \int_1^x \ln 5\,dt$

$= \ln 5 \cdot t\Big|_1^x = (x-1)\ln 5.$ $\Rightarrow$

Cum $\ln 5 > 0 \Rightarrow \lim\limits_{x\to+\infty} (x-1)\ln 5 = +\infty$

$\Rightarrow \lim\limits_{x\to+\infty} \int_1^x \ln f(t)\,dt = +\infty.$

Ex: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = (x^2+1)e^{-x}$

a) $\int_0^1 (e^x f(x) - 2)\,dx = -\frac{2}{3}$

b) $\int_1^e f(\ln x)\,dx = ?$

c) $I_n = \int_0^1 x^n f(x)\,dx$, $\forall n \in \mathbb{N}^*$. Calculați $\lim_{n\to\infty} I_n$.

Sol:

a) $\int_0^1 (x^2+1-2)\,dx = \int_0^1 (x^2-1)\,dx = \frac{x^3}{3}\Big|_0^1 - x\Big|_0^1 = \frac{1}{3} - 1 = -\frac{2}{3}$.

b) $\int_1^e ((\ln x)^2 + 1)\underbrace{e^{-\ln x}}_{\frac{1}{e^{\ln x}} = \frac{1}{x}}\,dx = \int_1^e ((\ln x)^2+1)\frac{dx}{x}$

$= \int_1^e \frac{dx}{x} + \int_1^e (\ln x)^2 \frac{dx}{x}$

$\ln x = t \Rightarrow \frac{dx}{x} = dt$

$\ln e = 1$

$= \ln x\Big|_1^e + \int_0^{\ln e} t^2\,dt$

$= \ln e + \frac{t^3}{3}\Big|_0^{\ln e} = 1 + \frac{1}{3} = \frac{4}{3}$.

c) pt orice $x \in [0,1]$,

$0 \le 1+x^2 \le 2$

$0 \le e^{-x} \le e^0 = 1$.

$0 \le f(x) \le 2$.

Atunci

$0 \le |I_n| = \left|\int_0^1 x^n f(x)\,dx\right|$

$\le \int_0^1 |x^n|\underbrace{|f(x)|}_{\le 2}\,dx$

$\le 2\int_0^1 x^n\,dx$

$= 2\cdot \frac{x^{n+1}}{n+1}\Big|_0^1$

$= \frac{2}{n+1}$.

Pentru $n \to +\infty \Rightarrow I_n \to 0$.

Ex: $f:(0,+\infty)\to\mathbb{R}$, $f(x)=x^3-\ln x$.

a) $\int_1^{\sqrt{2}} (f(x)+\ln x)\,dx = \frac{3}{4}$

b) $\int_1^{e} x(x^3-f(x))\,dx = ?$

c) $\int_1^{e^2} \frac{1}{x} f(\sqrt{x})\,dx = \frac{2e^3-5}{3}$.

Sol:

a) $\int_1^{\sqrt{2}} x^3\,dx = \frac{x^4}{4}\Big|_1^{\sqrt{2}} = \frac{4-1}{4} = \frac{3}{4}$.

b) $\int_1^{e} x\ln x\,dx = \int_1^{e} \left(\frac{x^2}{2}\right)'\ln x\,dx$

$= \frac{x^2}{2}\ln x\Big|_1^{e} - \int_1^{e} \frac{x^2}{2}\cdot\frac{1}{x}\,dx$

$= \frac{e^2}{2} - \frac{1}{2}\int_1^{e} x\,dx$

$= \frac{e^2}{2} - \frac{1}{2}\cdot\frac{x^2}{2}\Big|_1^{e} = \frac{e^2}{2} - \frac{1}{2}\left(\frac{e^2-1}{2}\right)$

$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4}$.

c) $\int_1^{e^2} \frac{1}{x}\left(x\sqrt{x} - \frac{1}{2}\ln x\right)dx$

$= \int_1^{e^2}\left(\sqrt{x} - \frac{1}{2}\cdot\underbrace{\frac{1}{x}}_{(\ln x)'}\ln x\right)dx$

$= \frac{2}{3}x^{3/2}\Big|_1^{e^2} - \frac{1}{2}\frac{(\ln x)^2}{2}\Big|_1^{e^2}$

$= \frac{2}{3}(e^3-1) - \frac{1}{4}\left((\ln e^2)^2 - (\ln 1)^2\right)$

$= \frac{2}{3}(e^3-1) - \frac{1}{4}\cdot 4$

$= \frac{2}{3}e^3 - \frac{2}{3} - 1$

$= \frac{2e^3-5}{3}$.

Ex: $f:\mathbb{R}\to\mathbb{R}$, $f(x)=2x-1$.

a) $\int_0^1 f(x)\,dx=0$

b) $\int_0^1 e^x|f(x)|\,dx=?$

c) $I_n=\int_0^1 f^n(x)\,dx$, $n\in\mathbb{N}^*$. Calculați $\lim\limits_{n\to\infty} I_n$.

Sol:

a) $\int_0^1 (2x-1)\,dx = x^2\Big|_0^1 - x\Big|_0^1 = 1-1=0.$

b) $|2x-1| = \begin{cases} 1-2x, & x<\frac{1}{2} \\ 2x-1, & x\geq \frac{1}{2}\end{cases}$ $\quad 2x-1=0 \Leftrightarrow x=\frac{1}{2}$

$$\int_0^1 e^x|2x-1|\,dx = \int_0^{1/2} e^x(1-2x)\,dx + \int_{1/2}^1 e^x(2x-1)\,dx$$
$$= \int_0^{1/2}(e^x)'(1-2x)\,dx + \int_{1/2}^1 (e^x)'(2x-1)\,dx$$
$$= e^x(1-2x)\Big|_0^{1/2} + 2\int_0^{1/2} e^x\,dx + e^x(2x-1)\Big|_{1/2}^1 - 2\int_{1/2}^1 e^x\,dx$$
$$= -1 + 2e^x\Big|_0^{1/2} + e - 2e^x\Big|_{1/2}^1$$
$$= -1 + 2\sqrt{e} - 2 + e - 2(e-\sqrt{e})$$
$$= 4\sqrt{e} - 3 - e.$$

c) $I_n = \int_0^1 (2x-1)^n\,dx \qquad 2x-1=t, \quad dx=\frac{dt}{2}$

$$= \int_{-1}^1 t^n \frac{dt}{2}$$
$$= \frac{1}{2}\,\frac{t^{n+1}}{n+1}\Big|_{-1}^1 = \frac{1}{2}\,\frac{1-(-1)^{n+1}}{n+1}.$$

$$0\leq |I_n| = \frac{1}{2}\,\frac{|1-(-1)^{n+1}|}{n+1}$$
$$\leq \frac{1}{2}\,\frac{|1|+|(-1)^{n+1}|}{n+1}$$
$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{n+1}$$
$$= \frac{1}{n+1} \longrightarrow 0 \text{ pt } n\to+\infty.$$

Clește $\Rightarrow I_n \to 0$.

Ex: $f:(-1,+\infty)\to\mathbb{R}$, $f(x)=\sqrt{x+1}$.

a) $\int_0^2 f^2(x)\,dx=4$.

b) $\int_0^1 \ln f(x)\,dx=?$

c) $\exists!\ x\in[0,+\infty)$ astfel încât $\int_0^x e^{f(t)}\,dt=2021$.

Sol:

a) $\int_0^2 (x+1)\,dx=\frac{x^2}{2}\Big|_0^2+x\Big|_0^2=2+2=4$.

b) $\int_0^1 \ln\sqrt{x+1}\,dx=\int_0^1 \ln(x+1)^{\frac{1}{2}}\,dx$

$=\frac{1}{2}\int_0^1 \ln(x+1)\,dx=\frac{1}{2}\int_0^1 (1+x)'\ln(1+x)\,dx$

$=\frac{1}{2}\left\{(1+x)\ln(1+x)\Big|_0^1-\int_0^1 (1+x)\cdot\frac{1}{1+x}\,dx\right]$

$=\frac{1}{2}\left\{2\ln 2-\int_0^1 dx\right]$

$=\frac{1}{2}\left\{2\ln 2-x\Big|_0^1\right]=\frac{1}{2}(2\ln 2-1)=\ln 2-\frac{1}{2}$

c) $F:(-1,+\infty)\to\mathbb{R}$, $F(x)=\int_0^x e^{\sqrt{t+1}}\,dt$

este o primitivă pentru f cu $F(0)=0$.

Cum $F'(x)=e^{\sqrt{x+1}}>0$ pe $(-1,+\infty)$, avem că F este strict crescătoare pe $(-1,+\infty)$. În particular,

F strict crescătoare pe $[0,+\infty)$

Avem $F(0)=0$.

$F(x)=\int_0^x \underbrace{\overbrace{e^{\sqrt{1+t}}}^{\geq 0}}_{\geq 1}\,dt\geq\int_0^x dt$

$=x,\ \forall x\geq 0$

$\Rightarrow \lim_{x\to+\infty}F(x)=+\infty$

F cont pe $[0,+\infty)$

$\Rightarrow$

$\exists!\ x\in[0,+\infty)$ a.î. $F(x)=2021>0$.

Ex: $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\dfrac{2x-2}{x^2+4}$

a) $\int_0^2 (x^2+4)f(x)\,dx = 0$.

b) $\int_0^{2\sqrt{3}} f(x)\,dx = ?$

c) $\int_1^x f(t)\,dt \geq 0, \ \forall x \in \mathbb{R}$

Sol:

a) $\int_0^2 \cancel{(x^2+4)} \dfrac{2x-2}{\cancel{x^2+4}}\,dx = \int_0^2 (2x-2)\,dx = (x^2-2x)\Big|_0^2$
$= 4-4=0$

b) $\int_0^{2\sqrt{3}} \dfrac{2x-2}{x^2+4}\,dx = \int_0^{2\sqrt{3}} \dfrac{2x}{x^2+4} - 2\int_0^{2\sqrt{3}} \dfrac{dx}{x^2+4}$

$= \int_0^{2\sqrt{3}} \dfrac{(x^2+4)'}{x^2+4}\,dx - 2\int_0^{2\sqrt{3}} \dfrac{dx}{x^2+2^2}$

$= \ln(x^2+4)\Big|_0^{2\sqrt{3}} - \cancel{2}\cdot\dfrac{1}{\cancel{2}}\operatorname{arctg}\dfrac{x}{2}\Big|_0^{2\sqrt{3}}$

$= \ln 16 - \ln 4 - \operatorname{arctg}\sqrt{3}$

$= 4\ln 2 - 2\ln 2 - \dfrac{\pi}{3}$

$= 2\ln 2 - \dfrac{\pi}{3}$.

c) $F:\mathbb{R}\to\mathbb{R}$, $F(x)=\int_1^x f(t)\,dt$ este o primitivă pentru f pe $\mathbb{R}$. Adică

$F'(x) = \dfrac{2(x-1)}{x^2+4}, \ \forall x\in\mathbb{R}$.

Atunci

$F' \leq 0$ pe $(-\infty,1] \Rightarrow F \searrow$ pe $(-\infty,1]$

$F' \geq 0$ pe $[1,+\infty) \Rightarrow F \nearrow$ pe $[1,+\infty)$.

x	$-\infty$		1		$+\infty$
F'		$- - -$	0	$+ \ + \ + +$	
F		$\searrow$	0	$\nearrow$	

$F(1)=0$.

$F(x) \geq F(1) = 0, \ \forall x\in\mathbb{R} \Rightarrow$

$\int_1^x f(t)\,dt \geq 0, \ \forall x\in\mathbb{R}$.

Ex: $f: \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2+1$

a) $\int_0^1 f(x)\,dx = \frac{4}{3}$

b) $\int_0^1 e^x f(x)\,dx = ?$

c) $\int_{-1}^1 |x \ln f(x)|\,dx = 2\ln 2 - 1.$

Sol: a) $\int_0^1 (x^2+1)\,dx = \frac{x^3}{3}\Big|_0^1 + x\Big|_0^1 = \frac{1}{3} + 1 = \frac{4}{3}$

b) $\int_0^1 e^x(x^2+1)\,dx = \int_0^1 (e^x)'(x^2+1)\,dx = e^x(x^2+1)\Big|_0^1 - \int_0^1 e^x (x^2+1)'\,dx$

$= 2e-1 - \int_0^1 e^x \cdot 2x\,dx = 2e-1-2\int_0^1 (e^x)' \cdot x\,dx$

$= 2e-1-2\left(e^x x\Big|_0^1 - \int_0^1 e^x \cdot 1\,dx\right)$

$= 2e-1-2\left(e - e^x\Big|_0^1\right) = 2e-1-2(e-(e-1))$

$= 2e-1-2 = 2e-3.$

c) $\int_{-1}^1 |x \ln(1+x^2)|\,dx = \int_0^1 |x| \cdot |\ln\underbrace{(1+x^2)}_{\geq 1}|\,dx + \int_{-1}^0 |x| \cdot |\underbrace{\ln(1+x^2)}_{\geq 0}|\,dx$

$= \int_0^1 x \ln(1+x^2)\,dx - \int_{-1}^0 x \ln(1+x^2)\,dx$

$x = -t$

$dx = -dt$

$= \int_0^1 x \ln(1+x^2)\,dx + \int_0^1 t \ln(1+t^2)\,dt$

$= 2\int_0^1 x \ln(1+x^2)\,dx$

$= \int_0^1 (x^2+1)' \ln(1+x^2)\,dx$

$= \left[(x^2+1)\ln(1+x^2)\Big|_0^1 - \int_0^1 \cancel{(x^2+1)} \frac{2x}{\cancel{1+x^2}}\,dx\right]$

$= \left[2\ln 2 - \int_0^1 2x\,dx\right]$

$= 2\ln 2 - x^2\Big|_0^1$

$= 2\ln 2 - 1.$